

THE MOMENTUM PRESSURE DIFFERENCE UNDER  
ARBITRARY ACCOMMODATION AT A SURFACE

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The phenomenon of thermomolecular pressure difference is described in terms of the solution to the linearized Bubnov-Galerkin-Knudsen model, assuming an arbitrary tangential-momentum accommodation coefficient for molecules impinging on a surface. A comparison of theoretical results with test data shows a close agreement between them.

An analysis of available test data indicates that the phenomenon of thermomolecular pressure difference depends on the tangential-momentum accommodation coefficient for gas molecules impinging on a surface [1-3]. Therefore, a comparative evaluation of theoretical results and test data over the entire range of Knudsen number values must correctly account for the accommodating capability of the channel surface with respect to various gases.

Let the gas in a circular capillary be perturbed by a temperature gradient in the z-direction. Under steady conditions there will appear a corresponding pressure gradient which reduces to zero the mass flow rate of gas through any section of the capillary. We assume that, inasmuch as the perturbation is small, the velocity distribution of molecules approaches a Maxwellian one:

$$f(c, c_z, r, z) = f_0 [1 + c_z \varphi(c, r)],$$

with  $r$  denoting the two-dimensional radius vector in a plane normal to the z-axis and  $c, c_z$  denoting respectively the dimensionless r- and z- component of the molecular velocity.

A linearization of this problem makes it feasible to analyze the effects due to the pressure gradient and those due to the temperature gradient independently [4-8]:

$$c \frac{\partial \varphi_P}{\partial r} + v = -\delta [\varphi_P - 2u_P], \quad (2)$$

$$c \frac{\partial \varphi_T}{\partial r} + \tau \left( c^2 + c_z^2 - \frac{5}{2} \right) = -\delta [\varphi_T - 2u_T], \quad (3)$$

$$v = \frac{R}{P} \frac{dP}{dz}, \quad \tau = \frac{R}{T} \frac{dT}{dz}.$$

From Eqs. (2) and (3) one can easily determine the distribution function of molecules at any point with the coordinate  $s$  which coincides with some initially selected direction of velocity  $c$  (Fig. 1):

$$\begin{aligned} \varphi_P(s, c) = & \exp\left(-\frac{\delta}{c}s\right) \left\{ \varphi_P(0, c) \right. \\ & \left. + \int_0^s \frac{1}{c} \exp\left(\frac{\delta}{c}s'\right) (2\delta u_P - v) ds' \right\}, \end{aligned} \quad (4)$$

$$\begin{aligned} \varphi_T(s, c) = & \exp\left(-\frac{\delta}{c}s\right) \left\{ \varphi_T(0, c) \right. \\ & \left. + \int_0^s \frac{1}{c} \exp\left(\frac{\delta}{c}s'\right) \left[ 2\delta u_T - \tau \left( c^2 + c_z^2 - \frac{5}{2} \right) \right] ds' \right\}, \end{aligned} \quad (5)$$

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where  $\varphi_P(0, c)$  and  $\varphi_T(0, c)$  define the corresponding distribution functions of molecules emitted from point M on the capillary surface in the direction of  $c$ .

The boundary conditions and the conditions of symmetry for this problem are

$$\varphi_i(0, c) = (1 - \varepsilon) \varphi_i(0, -c), \quad (6)$$

$$\varphi_i(0, -c) = \varphi_i(l, c) \quad (i = P, T). \quad (7)$$

Here  $\varepsilon$  is the tangential-momentum accommodation coefficient for molecules impinging on the surface, and  $l = 2(1 - r^2 \sin^2 \alpha)^{1/2}$  is the length of chord MN.

From Eq. (4), with (6) and (7) taken into account, we obtain

$$\begin{aligned} \varphi_P(0, c) &= \frac{(1 - \varepsilon) \exp\left(-\frac{\delta}{c} l\right)}{1 - (1 - \varepsilon) \exp\left(-\frac{\delta}{c} l\right)} \\ &\times \int_0^l \frac{1}{c} (2\delta u_P - v) \exp\left(\frac{\delta}{c} s'\right) ds'. \end{aligned} \quad (8)$$

It can be shown that  $\varphi_T(0, c) \equiv 0$  for any value of parameter  $\delta$ . The proof will be omitted here, because of rather unwieldy though not very complex calculations. An analogous result has been obtained in [9], where the velocity of sliding due to heat is shown to be independent of the tangential-momentum accommodation coefficient.

The definition of macroscopic velocities  $u_P$  and  $u_T$  representing all possible directions  $c$  and surface points M, with the aid of expressions (4), (5), and (8), yields the following linear nonhomogeneous integral equations:

$$u_P = \frac{v}{2\delta} (1 - \psi_P), \quad u_T = -\frac{\tau}{2\delta} (1 - \psi_T),$$

$$\psi_P = 1 + \frac{\delta}{\pi} \iint_{(\Sigma)} \psi_P \frac{K(r, r')}{|r - r'|} dr', \quad (9)$$

$$\psi_T = 1 - \frac{\delta}{\pi} \iint_{(\Sigma)} \frac{J_2(\delta |r - r'|)}{|r - r'|} dr' + \frac{\delta}{\pi} \iint_{(\Sigma)} \psi_T \frac{J_0(\delta |r - r'|)}{|r - r'|} dr',$$

$$K(r, r') = \int_0^\infty \frac{1 + (1 - \varepsilon) \exp\left(-\frac{\delta}{c} l\right)}{1 - (1 - \varepsilon) \exp\left(-\frac{\delta}{c} l\right)} \exp\left(-c^2 - \frac{\delta}{c} |r - r'|\right) dc, \quad (10)$$

$$J_n(x) = \int_0^\infty v^n \exp\left(-v^2 - \frac{x}{v}\right) dv.$$

Integration in (9) and (10) is performed with respect to the cross section area of the capillary  $\Sigma$ .

One may apply the Bubnov-Galerkin process [10] to Eqs. (9) and (10), for the purpose of which  $\psi_P$  and  $\psi_T$  are approximated by series in even basis functions  $\{r^{2k}\}$ :

$$\psi_P(r, \varepsilon) = \sum_{k=0}^{\infty} A_k(\varepsilon) r^{2k}, \quad \psi_T(r, \varepsilon) = \sum_{k=0}^{\infty} B_k(\varepsilon) r^{2k}, \quad (11)$$

while coefficients  $A_k$  and  $B_k$  are determined from the stipulation that Eqs. (9) and (10) be orthogonal to the respective basis functions. The fast convergence of this method makes it feasible to consider two terms of expansions (11) only.

The referred rates of gas flow due to the pressure gradient  $Q_P$  and due to the temperature gradient  $Q_T$  are determined according to the respective formulas

$$Q_P(\delta; \varepsilon) = \frac{2 \langle u_P \rangle}{\Sigma v} = \frac{1}{\delta} \left[ 1 - \frac{\pi(0.25a_{22} - a_{12} + a_{11})}{a_{11}a_{22} - a_{12}^2} \right], \quad (12)$$

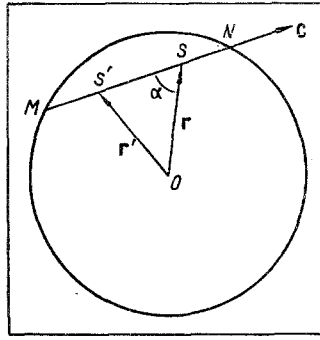


Fig. 1. Geometrical diagram for Eqs. (4) and (5).

$$Q_T(\delta; \varepsilon) = \frac{2 \langle u_T \rangle}{\Sigma \tau} = \frac{1}{\delta} \left[ -1 + \frac{0.5(b_1 b_{22} - b_2 b_{12}) + b_2 b_{11} - b_1 b_{12}}{b_{11} b_{22} - b_{12}^2} \right], \quad (13)$$

where the symbols inside  $\langle \rangle$  brackets denote mean-over-the section macroscopic velocities of the gas. Formulas for calculating the coefficients  $a$  and  $b$  are derived in the Appendix.

As has been noted earlier, no macroscopic flow through a capillary occurs under steady conditions:

$$\langle u_T \rangle + \langle u_P \rangle = 0, \quad (14)$$

and, therefore,

$$\frac{dP}{P} = \gamma(\delta; \varepsilon) \frac{dT}{T}, \quad \gamma(\delta; \varepsilon) = \frac{Q_T(\delta)}{Q_P(\delta; \varepsilon)}. \quad (15)$$

When the perturbation of a gas from its steady state is small ( $\Delta T/\bar{T} < 1$ ), then the tangential-momentum accommodation coefficient  $\varepsilon$  may be assumed constant along the capillary and to correspond to the mean surface temperature. Integration of (15) along the capillary will then yield a formula for the thermomolecular pressure difference:

$$\frac{P_1}{P_2} = \left( \frac{T_1}{T_2} \right)^\gamma. \quad (16)$$

In the limiting case of a free-molecular mode ( $\delta = 0$ ), relation (16) yields the Knudsen equation with a correction accounting for the imperfect tangential-momentum accommodation of gas molecules at a surface:

$$\frac{P_1}{P_2} = \left( \frac{T_1}{T_2} \right)^{\frac{1}{2} \left( \frac{\varepsilon}{2-\varepsilon} \right)}. \quad (17)$$

The theoretical results are shown in Fig. 2 together with test data obtained with long glass capillaries at temperatures  $T_1 = 77^\circ\text{K}$  and  $T_2 = 299^\circ\text{K}$  [3]. According to the diagram, the theoretical curves for  $\varepsilon = 1$  (1),  $\varepsilon = 0.96$  (2), and  $\varepsilon = 0.92$  (3) do accurately enough describe the test data for argon, hydrogen, and neon respectively (the maximum discrepancy does not exceed 2%).

Thus, the satisfactory agreement between theoretical and experimental values lends support to this theory as a basis for calculating the corrections to gas pressure measurements made with instruments at temperatures other than the test temperature, also for determining the tangential-momentum accommodation coefficient for molecules of various gases impinging on a surface.

#### APPENDIX

The coefficients in Eqs. (12) and (13) are determined with the aid of the following formulas:

$$\begin{aligned} a_{11} &= b_{11} - c_{11}, \quad a_{12} = b_{12} - c_{12}, \quad a_{22} = b_{22} - c_{22}, \\ b_{11} &= -\frac{\pi}{\delta^2} - \frac{10\sqrt{\pi}}{\delta^3} + 8 \left\{ I_1^0(\delta) + \frac{6}{\delta^2} I_3^0(\delta) \right. \\ &\quad \left. + \left( \frac{4}{\delta} + \frac{15}{\delta^3} \right) I_2^1(\delta) + \frac{10}{\delta^2} I_1^2(\delta) \right\}; \quad b_{22} = 8I_1^0(\delta); \\ b_{12} &= 8 \left\{ I_1^0(\delta) + \frac{1}{\delta^2} \left[ I_3^0(\delta) + 2\delta I_2^1(\delta) - \frac{\pi}{8} \right] \right\}; \quad b_2 = 8I_3^0(\delta); \end{aligned}$$

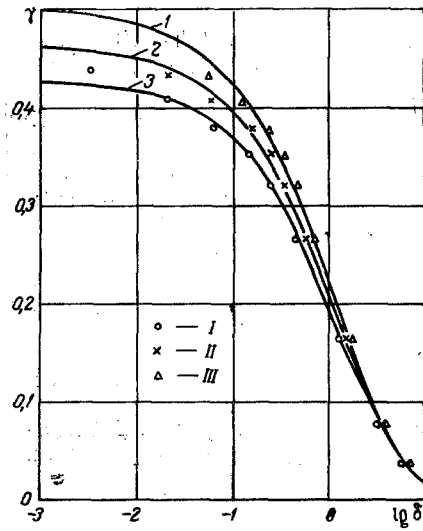


Fig. 2. Exponent  $\gamma$  as a function of the dimensionless collision parameter  $\delta$  and of the tangen-momentum accommodation coefficient  $\varepsilon = 1.0$  (1),  $0.96$  (2),  $0.92$  (3) (theoretical curves), test points for neon (I), hydrogen (II), argon (III).

$$b_1 = 8 \left\{ I_3^0(\delta) + \frac{1}{\delta^2} \left[ I_5^0(\delta) + 2\delta I_4^1(\delta) - \frac{\pi}{4} \right] \right\};$$

$$c_{22} = -8 \sum_{j=1}^{\infty} (1-\varepsilon)^j \left\{ (j-1) I_1^0[(j-1)\delta] - 2j I_1^0(j\delta) \right.$$

$$\left. + (j+1) I_1^0[(j+1)\delta] \right\};$$

$$c_{12} = c_{22} - \frac{8}{\delta} \sum_{j=1}^{\infty} (1-\varepsilon)^j \left\{ 2(j+1) I_2^1[j \right.$$

$$\left. + 1)\delta] - 2(j-1) I_2^1[(j-1)\delta] + \frac{(2j-1)}{\delta} I_3^0[(j-1)\delta] \right.$$

$$\left. - \frac{4j}{\delta} I_3^0(j\delta) + \frac{(2j+1)}{\delta} I_3^0[(j+1)\delta] \right\};$$

$$c_{11} = c_{12} - \frac{8}{\delta} \sum_{j=1}^{\infty} (1-\varepsilon)^j \left\{ 2(j+1) I_2^1[(j+1)\delta] \right.$$

$$\left. - 2(j-1) I_2^1[(j-1)\delta] + \frac{(6j-5)}{\delta} I_3^0[(j-1)\delta] \right.$$

$$\left. + \frac{4j}{\delta} I_3^0(j\delta) + \frac{(6j+5)}{\delta} I_3^0[(j+1)\delta] - \frac{3\sqrt{\pi}}{\delta^2} \right.$$

$$\left. - \frac{2(4j-5)}{\delta^2} I_4^1[(j-1)\delta] + \frac{4}{\delta^2} I_4^1(j\delta) + \frac{2(4j+5)}{\delta^2} I_4^1[(j+1)\delta] \right.$$

$$\left. + \frac{4j}{\delta^3} I_5^0[(j-1)\delta] - \frac{8j}{\delta^3} I_5^0(j\delta) + \frac{4j}{\delta^3} I_5^0[(j+1)\delta] \right\}.$$

For evaluating the integrals

$$I_m^n(\delta) = \int_0^1 v^n (1-v^2)^{1/2} J_m(2\delta v) dv$$

one can use asymptotic expansions of functions  $J_m(x)$  [11]. It is important to know the first four integrals  $I_1^0$ ,  $I_1^1$ ,  $I_2^0$ , and  $I_2^1$ , the other integrals can be calculated according to the recurrence formula

$$2I_m^n(\delta) = (m-1) I_{m-2}^n(\delta) + 2\delta I_{m-3}^{n+1}(\delta).$$

#### NOTATION

$c_i$  is the  $i$ -th component of dimensionless velocity;

$u_P$  is the macroscopic velocity due to pressure gradient;  
 $u_T$  is the macroscopic velocity due to temperature gradient;  
 $\delta$  is the dimensionless collision parameter;  
 $P$  is the pressure;  
 $T$  is the temperature;  
 $R$  is the radius of capillary tube;  
 $\varepsilon$  is the tangential-momentum accommodation coefficient;  
 $\Sigma$  is the cross section area of capillary tube;  
 $\gamma$  is the exponent of thermomolecular pressure difference;  
 $Q_P$  is the referred flow rate due to pressure gradient;  
 $Q_T$  is the referred flow rate due to temperature gradient.

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